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An Ikehara-type theorem for functions convergent to zero

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Abstract

We establish an analogue of the Ikehara theorem for positive non-increasing functions convergent to zero. In particular, this provides a complete proof to the results formulated in Diekmann, Kaper (1978) [5] and Carr, Chmaj (2004) [1], which are widely used nowadays to prove the uniqueness of traveling waves for various reaction-diffusion equations.

Keywords: Ikehara theorem, complex Tauberian theorem, Laplace transform, asymptotic behavior, traveling waves

2010 Mathematics Subject Classification: 40E05, 44A10, 35B40

1 Introduction

The Ikehara theorem and its extensions are the so-called complex Tauberian theorems, inspired, in particular, by the number theory, see e.g. the review [12]. The following version of the Ikehara theorem can be found in [8, Subsection 2.5.7]:

Theorem 1. *Let ϕ be a positive monotone increasing function, and let there exist $\mu > 0$, $j > 0$, such that*

$$\int_0^\infty e^{-tz} \phi(t) dt = \frac{F(z)}{(z - \mu)^j}, \quad \operatorname{Re} z > \mu,$$

where F is holomorphic on $\{\operatorname{Re} z \geq \mu\}$. Then,

$$\phi(t) \sim \frac{F(\mu)}{\Gamma(j)} t^{j-1} e^{\mu t}, \quad t \rightarrow \infty.$$

The case of $j = 1$ goes back to [11] and the general case was firstly considered in [4]. For more recent results in the case $j = 1$ see e.g. [3, 14]. Alternatively, Theorem 1 may be formulated for the Stieltjes measure $d\phi(t)$ instead of $\phi(t)dt$ obtaining similar asymptotics for ϕ (see e.g. Proposition 2.1 below).

In Theorem 1, ϕ increases to ∞ . In [5, Lemma 6.1] (for $j = 1$) and in [1, Proposition 2.3] (for $j > 0$), the similar results were stated for positive monotone

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decreasing φ (cf., correspondingly, Propositions 2.4 and 2.2 below). The aim of both generalizations was to find an *a priori* asymptotics for solutions to a class of nonlinear integral equations. In [1], in particular, it was applied to the study of the uniqueness of traveling wave solutions to certain nonlocal reaction-diffusion equations; see also e.g. [2, 6, 7, 13, 16, 17, 19]. Note also that then the case $j = 2$ corresponded to the traveling wave with the minimal speed.

In both papers [1, 5] no proof was given, mentioning that it is supposed to be analogous to the case of increasing ϕ without any further details. In Theorem 2 below, we prove an analogue of Theorem 1 for non-increasing function, and in Proposition 2.2 we apply it to prove the mentioned result of [1]. We require, however, an *a priori* regular decaying of φ , namely, we assume that there exists $\nu > 0$, such that $\varphi(t)e^{\nu t}$ is an increasing function. We require also the convergence of $\int_0^\infty e^{zt}d\varphi(t)$ for $0 < \operatorname{Re} z < \mu$ instead of the weaker corresponding assumption for $\int_0^\infty e^{zt}\varphi(t)dt$.

Beside the aim to present a proof, the reason for the generalization we provide was to omit the requirement on the function F to be analytical on the line $\{\operatorname{Re} z = \mu\}$ keeping the general case $j > 0$. We were motivated by the integro-differential equation we studied in [10] (which covers the equations considered in [1]), where the Laplace-type transform of the traveling wave with the minimal speed (that requires, recall, $j = 2$) might be not analytical at $z = \mu$.

Our result is based on a version of the Ikehara-Ingham theorem proposed in [15], see Proposition 2.1 below. Using the latter result, we prove also in Proposition 2.4 a generalization of [5, Lemma 6.1] (under the regularity assumptions on φ mentioned above).

2 Main results

Let, for any $D \subset \mathbb{C}$, $\mathcal{H}(D)$ be the class of all holomorphic functions on D .

Theorem 2. *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ := [0, \infty)$ be a non-increasing function such that, for some $\mu > 0$, $\nu > 0$,*

$$\text{the function } e^{\nu t}\varphi(t) \text{ is non-decreasing,} \quad (2.1)$$

and

$$\int_0^\infty e^{zt}d\varphi(t) < \infty, \quad 0 < \operatorname{Re} z < \mu. \quad (2.2)$$

Let also the following assumptions hold.

1. *There exist a constant $j > 0$ and complex-valued functions*

$$H \in \mathcal{H}(0 < \operatorname{Re} z \leq \mu), \quad F \in \mathcal{H}(0 < \operatorname{Re} z < \mu) \cap C(0 < \operatorname{Re} z \leq \mu),$$

such that the following representation holds

$$\int_0^\infty e^{zt}\varphi(t)dt = \frac{F(z)}{(\mu - z)^j} + H(z), \quad 0 < \operatorname{Re} z < \mu. \quad (2.3)$$

2. For any $T > 0$,

$$\lim_{\sigma \rightarrow 0+} g_j(\sigma) \sup_{|\tau| \leq T} |F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)| = 0, \quad (2.4)$$

where, for $\sigma > 0$,

$$g_j(\sigma) := \begin{cases} \sigma^{j-1}, & 0 < j < 1, \\ \log \sigma, & j = 1, \\ 1, & j > 1. \end{cases} \quad (2.5)$$

Then φ has the following asymptotic

$$\varphi(t) \sim \frac{F(\mu)}{\Gamma(j)} t^{j-1} e^{-\mu t}, \quad t \rightarrow \infty. \quad (2.6)$$

The proof of Theorem 2 is based on the following result due to Tenenbaum.

Proposition 2.1 (“Effective” Ikehara–Ingham Theorem, cf. [15, Theorem 7.5.11]).
Let $\alpha(t)$ be a non-decreasing function such that, for some fixed $a > 0$, the following integral converges:

$$\int_0^\infty e^{-zt} d\alpha(t), \quad \operatorname{Re} z > a. \quad (2.7)$$

Let also there exist constants $D \geq 0$ and $j > 0$, such that for the functions

$$G(z) := \frac{1}{a+z} \int_0^\infty e^{-(a+z)t} d\alpha(t) - \frac{D}{z^j}, \quad \operatorname{Re} z > 0, \quad (2.8)$$

$$\eta(\sigma, T) := \sigma^{j-1} \int_{-T}^T |G(2\sigma + i\tau) - G(\sigma + i\tau)| d\tau, \quad T > 0, \quad (2.9)$$

one has that

$$\lim_{\sigma \rightarrow 0+} \eta(\sigma, T) = 0, \quad T > 0. \quad (2.10)$$

Then

$$\alpha(t) = \left\{ \frac{D}{\Gamma(j)} + O(\rho(t)) \right\} e^{at} t^{j-1}, \quad t \geq 1, \quad (2.11)$$

where

$$\rho(t) := \inf_{T \geq 32(a+1)} \left\{ T^{-1} + \eta(t^{-1}, T) + (Tt)^{-j} \right\}. \quad (2.12)$$

Proof of Theorem 2. We first express $\int_0^\infty e^{\lambda t} \varphi(t) dt$ in the form (2.7). Fix any $a > 0$ such that $\mu + a > \nu$. Then, by (2.1), the function

$$\alpha(t) := e^{(\mu+a)t} \varphi(t), \quad t > 0, \quad (2.13)$$

is increasing. Since φ is monotone, then, for any $0 < \operatorname{Re} z < \mu$, one has

$$\int_0^\infty e^{-(a+z)t} d\alpha(t) = (\mu + a) \int_0^\infty e^{(\mu-z)t} \varphi(t) dt + \int_0^\infty e^{(\mu-z)t} d\varphi(t), \quad (2.14)$$

where both integrals in the right hand side of (2.14) converge, for $0 < \operatorname{Re} z < \mu$, because of (2.2)–(2.3).

Then, by [18, Corollary II.1.1a], the integral in the left hand side of (2.14) converges, for *all* $\operatorname{Re} z > 0$. Therefore, by [18, Theorem II.2.3a], one gets another representation for the latter integral, for $\operatorname{Re} z > 0$:

$$\int_0^\infty e^{-(a+z)t} d\alpha(t) = -\varphi(0) + (a+z) \int_0^\infty e^{(\mu-z)t} \varphi(t) dt. \quad (2.15)$$

Let G be given by (2.8) with $\alpha(t)$ as above and $D := F(\mu)$. Combining (2.15) with (2.3) (where we replace z by $\mu - z$), we obtain, for $0 < \operatorname{Re} z < \mu$,

$$G(z) = \frac{F(\mu - z) - F(\mu)}{z^j} + K(z),$$

$$K(z) := H(\mu - z) - \frac{\varphi(0)}{a + z}.$$

We check the condition (2.10); one can assume, clearly, that $0 < \sigma < \frac{\mu}{2}$. Since $K \in \mathcal{H}(0 \leq \operatorname{Re} z < \mu)$, one easily gets that

$$\begin{aligned} & \lim_{\sigma \rightarrow 0+} \sigma^{j-1} \int_{-T}^T |G(2\sigma + i\tau) - G(\sigma + i\tau)| d\tau \\ & \leq \lim_{\sigma \rightarrow 0+} \sigma^{j-1} \int_{-T}^T \left| \frac{F(\mu - 2\sigma - i\tau) - F(\mu)}{(2\sigma + i\tau)^j} - \frac{F(\mu - \sigma - i\tau) - F(\mu)}{(\sigma + i\tau)^j} \right| d\tau \\ & \leq \lim_{\sigma \rightarrow 0+} \sigma^{j-1} \int_{-T}^T \left| \frac{F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)}{(\sigma + i\tau)^j} \right| d\tau \\ & \quad + \lim_{\sigma \rightarrow 0+} \sigma^{j-1} \int_{-T}^T |F(\mu - 2\sigma - i\tau) - F(\mu)| \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| d\tau, \\ & =: \lim_{\sigma \rightarrow 0+} A_j(\sigma) + \lim_{\sigma \rightarrow 0+} B_j(\sigma). \end{aligned} \quad (2.16)$$

We now prove that both limits in (2.16) are equal to 0. For each $j > 0$, we define the function

$$h_j(\sigma) := \sigma^{j-1} \int_{-T}^T \frac{1}{(\sigma^2 + \tau^2)^{\frac{j}{2}}} d\tau, \quad \sigma > 0. \quad (2.17)$$

We have then

$$A_j(\sigma) \leq \sup_{|\tau| \leq T} |F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)| h_j(\sigma). \quad (2.18)$$

It is straightforward to check that

$$h_1(\sigma) = 2 \log \frac{\sqrt{T^2 + \sigma^2} + T}{\sigma} \sim -2 \log \sigma, \quad \sigma \rightarrow 0+.$$

For $j \neq 1$, we make the substitution $\tau = \sigma u$ in (2.17), then

$$h_j(\sigma) = \int_{-T/\sigma}^{T/\sigma} \frac{du}{(1+u^2)^{j/2}}. \quad (2.19)$$

Then, for $j > 1$, $\lim_{\sigma \rightarrow 0} h_j(\sigma) < \infty$, since the corresponding integral is convergent, and we obtain $h_j(\sigma) = O(1)$, as $\sigma \rightarrow 0+$. For $0 < j < 1$,

$$h_j(\sigma) \leq O(1) + 2 \int_1^{T/\sigma} \frac{du}{u^j} = O(\sigma^{j-1}), \quad \sigma \rightarrow 0+.$$

Combining the results we have that

$$h_j(\sigma) = O(|g_j(\sigma)|), \quad \sigma \rightarrow 0+, \quad j > 0,$$

that, together with (2.18) and (2.4), yield $\lim_{\sigma \rightarrow 0+} A_j(\sigma) = 0$.

Take now an arbitrary $\beta \in (0, \mu)$ and consider, for each $T > 0$, the set

$$K_{\beta, \mu, T} := \{z \in \mathbb{C} \mid \beta \leq \operatorname{Re} z \leq \mu, \quad |\operatorname{Im} z| \leq T\}. \quad (2.20)$$

Let $0 < \sigma < \mu^2$; since $F \in C(K_{\sqrt{\sigma}, \mu, T})$, there exists $C_1 > 0$ such that $|F(z)| \leq C_1$, $z \in K_{\sqrt{\sigma}, \mu, T}$. Therefore,

$$\begin{aligned} B_j(\sigma) &\leq \sigma^{j-1} \sup_{|\tau| \leq \sqrt{\sigma}} |F(\mu - 2\sigma - i\tau) - F(\mu)| \int_{|\tau| \leq \sqrt{\sigma}} \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| d\tau \\ &\quad + 2C_1 \sigma^{j-1} \int_{\sqrt{\sigma} \leq |\tau| \leq T} \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| d\tau. \end{aligned} \quad (2.21)$$

Next, since

$$\begin{aligned} j \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| &= \left| \int_{\sigma + i\tau}^{2\sigma + i\tau} \frac{1}{z^{j+1}} dz \right| = \left| \int_0^1 \frac{\sigma}{((1+t)\sigma + i\tau)^{j+1}} dt \right| \\ &\leq \int_0^1 \frac{\sigma}{((1+t)^2 \sigma^2 + \tau^2)^{\frac{j+1}{2}}} dt \leq \frac{\sigma}{(\sigma^2 + \tau^2)^{\frac{j+1}{2}}}, \end{aligned}$$

we can continue (2.21) as follows, cf. (2.17),

$$\begin{aligned} j B_j(\sigma) &\leq \sup_{|\tau| \leq \sqrt{\sigma}} |F(\mu - 2\sigma - i\tau) - F(\mu)| h_{j+1}(\sigma) \\ &\quad + 4C_1 \int_{\sqrt{\sigma} \leq \tau \leq T} \frac{\sigma^j}{(\sigma^2 + \tau^2)^{\frac{j+1}{2}}} d\tau. \end{aligned} \quad (2.22)$$

By (2.19), functions h_{j+1} are bounded on $(0, \infty)$ for all $j > 0$. Next, since F is uniformly continuous on $K_{\sqrt{\sigma}, \mu, T}$, we have that, for any $\varepsilon > 0$ there exists

$\delta > 0$ such that $f(\mu, \sigma, \tau) := |F(\mu - 2\sigma - i\tau) - F(\mu)| < \varepsilon$, if only $4\sigma^2 + \tau^2 < \delta$. Therefore, if $\sigma > 0$ is such that $4\sigma^2 + \sigma < \delta$ then $\sup_{|\tau| \leq \sqrt{\sigma}} f(\mu, \sigma, \tau) < \varepsilon$ hence

$$\sup_{|\tau| \leq \sqrt{\sigma}} |F(\mu - 2\sigma - i\tau) - F(\mu)| h_{j+1}(\sigma) \rightarrow 0, \quad \sigma \rightarrow 0+. \quad (2.23)$$

Finally, making the substitution $\tau = \sigma u$ in the integral in (2.22), we obtain that it is equal to

$$I_j := \int_{1/\sqrt{\sigma}}^{T/\sigma} \frac{du}{(1+u^2)^{\frac{j+1}{2}}} \leq \int_{1/\sqrt{\sigma}}^{T/\sigma} \frac{du}{u^{j+1}} = O(\sigma^{j/2}), \quad \sigma \rightarrow 0+.$$

As a result, $I_j \rightarrow 0$ as $\sigma \rightarrow 0+$, that, together with (2.23) and (2.22), proves that $B_j(\sigma) \rightarrow 0$, $\sigma \rightarrow 0+$.

Combining this with $A_j(\sigma) \rightarrow 0$, one gets (2.10) from (2.16); and we can apply Proposition 2.1. Namely, by (2.11),

$$\varphi(t)e^{(\mu+a)t} = \left\{ \frac{D}{\Gamma(j)} + O(\rho(t)) \right\} e^{at} t^{j-1}, \quad t \rightarrow \infty.$$

By (2.10) and (2.12), $\rho(t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore,

$$\varphi(t)e^{(\mu+a)t} \sim \frac{D}{\Gamma(j)} e^{at} t^{j-1}, \quad t \rightarrow \infty,$$

that is equivalent to (2.6) and finishes the proof. \square

The following simple proposition shows that if F in (2.3) is holomorphic on the line $\{\operatorname{Re} z = \mu\}$, then (2.4) holds.

Proposition 2.2. *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-increasing function such that, for some $\mu > 0$, $\nu > 0$, (2.1)–(2.2) hold. Suppose also that there exist $j > 0$ and $F, H \in \mathcal{A}(0 < \operatorname{Re} z \leq \mu)$, such that (2.3) holds. Then φ has the asymptotic (2.6).*

Proof. Take any $\beta \in (0, \mu)$ and $T > 0$. Let $K_{\beta, \mu, T}$ be defined by (2.20). Since $F \in \mathcal{A}(0 < \operatorname{Re} z \leq \mu)$, then $F' \in C(K_{\beta, \mu, T})$, and hence F' is bounded on $K_{\beta, \mu, T}$. Then one can apply a mean-value-type theorem for complex-valued functions, see e.g. [9, Theorem 2.2], to get that, for some $K > 0$,

$$|F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)| \leq K|\sigma|, \quad 2\sigma < \mu - \beta,$$

that yields (2.4) for all $j > 0$, cf. (2.5). Hence we can apply Theorem 2. \square

Remark 2.3. Note that, for $F \in \mathcal{A}(0 < \operatorname{Re} z \leq \mu)$ in (2.3), the holomorphic function H is redundant there, as we always can replace $F(z)$ by a holomorphic function $F(z) + H(z)(\mu - z)^j$. Therefore, Proposition 2.2 corresponds to [1, Proposition 2.3].

Proposition 2.4. *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-increasing function such that, for some $\mu > 0$, $\nu > 0$, (2.1)–(2.2) hold. Suppose also that there exist $j \geq 1$, $D > 0$, and $h : \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$H(z) := \int_0^\infty e^{zt} \varphi(t) dt - \frac{D}{(\mu - z)^j} \rightarrow h(\operatorname{Im} z), \quad \operatorname{Re} z \rightarrow \mu-, \quad (2.24)$$

uniformly (in $\operatorname{Im} z$) on compact subsets of \mathbb{R} . Then the following asymptotic holds,

$$\varphi(t) \sim \frac{D}{\Gamma(j)} t^{j-1} e^{-\mu t}, \quad t \rightarrow \infty. \quad (2.25)$$

Proof. Let $a > \max\{0, \nu - \mu\}$ and $\alpha(t)$ be given by (2.13). Let G be given by (2.8). Similarly to the proof of Theorem 2, we will get from (2.15) and (2.24), that

$$G(z) = H(\mu - z) - \frac{\varphi(0)}{a + z}, \quad 0 < \operatorname{Re} z < \mu.$$

Next, (2.24) implies (2.10). Hence, by Lemma 2.1, (2.25) holds that fulfilled the proof. \square

Note that the result in [5, Lemma 6.1] corresponds to $j = 1$ in Proposition 2.4.

Remark 2.5. It is worth noting that, for the case $j > 1$, we have, by (2.9), that if G is bounded, then (2.10) holds. Therefore, in this case, it is enough to assume that H in (2.24) is bounded to conclude (2.25).

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